Teaching Bond Valuation: A Differential Approach Demonstrating Duration and Convexity

TeWahn Hahn, David Lange[1]

ABSTRACT

A traditional bond pricing scheme used in introductory finance texts is simple enough but not necessarily intuitive. The differential approach suggested here presents premiums (discounts) as coupon interest over- (under-) payments to make bond pricing dynamics more intuitive. The primary pedagogic benefit is the differential approach demonstrates the more sophisticated bond valuation concepts of duration and convexity.

Introduction

The basic bond valuation formula is traditionally presented as a straight forward discounted cash flow application. The relationship between required yield and price is generally stated as "Bond prices and yields are inversely related: as yields increase, bond prices fall; as yields fall, bond prices rise" (Bodie, Kane and Marcus 2009. pg. 514). A second bond pricing observation is "An increase in a bond's yield to maturity results in a smaller price change than a decrease in yield of equal magnitude" (Bodie, Kane and Marcus 2009. pg. 514). Both observations are attributable to Malkiel (1962).

A pedagogic difficulty is students may be able to solve for the bond price given a change in interest rates, but may not fully understand the financial dynamics for bond price premiums and discounts. Further, the differential impact of an increase (decrease) in yield to maturity on bond price may be even more unclear.

This paper first demonstrates the traditional bond valuation approach. Second, a differential bond price premium and discount conceptual approach is described. Third, the traditional and differential approaches are compared. Fourth, the differential approach pedagogic insight into the more sophisticated bond valuation duration and convexity concept is demonstrated. An appendix proves the differential approach is equivalent to the traditional approach equation.

Traditional Approach

Bond pricing is a relatively straight forward application of the Discounted Cash Flow (DCF) valuation method, and common among introductory finance text books (Berk and DeMarzo 2007, Block and Hirt 2006, Brealey, Myers and Marcus 2007, Brigham and Houston 2007, Ross, Westerfield and Jordan 2007, and Keown, Martin, Petty and Scott 2006).

A general bond pricing equation is

$$V_{\text{bond}} = \sum_{t=1}^{t=n} \frac{\text{Coupon}}{(1+y)^t} + \frac{\text{Face Value}}{(1+y)^n}. \quad (1)$$

where n = time to maturity,
y = required yield,
Coupon = coupon (interest) rate (= C) * Face Value,
and Face Value (= par value) = $1,000 in U.S.

[1] Auburn University Montgomery, Department of Accounting & Finance, P.O. Box 244023, Montgomery, AL 36124-4023

The first term is the sum of present values of future coupon payments or interest payments. The second term is the present value of the face value or the principal. The bond cash flows including the coupon interest payments, and the face value or principal are determined at issue and fixed, with the exception of floating rate bonds.

Equation (1) is simple, but not necessarily intuitive. Introductory finance texts generally refer to the fixed cash flows and the varying required yield in equation (1). As such, par bonds are defined as selling at face or par value when the required yield is equal to the coupon rate. Discount bonds are bonds selling below face value when the required yield is greater than the coupon rate. Premium bonds are bonds selling above face value when the required yield is below the coupon rate. Discount and premium bond prices deviate from the face value or principal because of the change in the required yield. The amount of the discount (premium) is found by subtracting the Face Value from the traditional formula Market Price.

Several authors have noticed the pedagogic limitations of the traditional approach. Boyles, Secrest, and Burney (2005) expanded the above traditional bond pricing formula to between coupon payment dates. Yang (2002) acknowledges the above textbook approach, and suggests the emphasis on the inverse relationship between yields and bond prices overlooks the impact of the loanable funds market on the behavior of bond prices. Our paper extends the traditional pedagogic approach by providing more intuition behind bond pricing.

Differential Approach

The differential approach calculates the bond price as equal to Face value of the bond plus the present value of under-paid or over-paid coupon payments from the investors' view. Under-paid coupon payments would exist when investors' perceived fair level of interest rate is higher than the coupon interest rate the bond is paying. In this case, the price of the bond would be lower than face value ($1,000) by exactly the present value of under-paid coupon interest payments over the life of the bond contract. Over-paid coupon payments would exist when investors' perceived fair level of interest rate is lower than the coupon interest rate the bond is paying. In this case, the price of the bond would be higher than face value ($1,000) by exactly the present value of over-paid coupon interest payments over the life of the bond contract. Needless to say, if investors' perceived fair level of interest is the same as the coupon interest rate the bond is paying, there will be no under-paid or over-paid coupon payments from investors' view and the bond price will be the same as face value ($1,000).

The deviations of market prices from face value, as noted above, can be found once one calculates the bond price using either a financial calculator or a spreadsheet program, following equation (1). However, determining the deviation using the resulting bond price according to equation (1) does not offer the same conceptual pedagogic advantages as the suggested differential approach.

Comparison of Traditional and Differential Conceptual Approach

Introductory finance texts often define a bond selling at a discount or at a premium, but no more. Introductory finance texts generally do not discuss how much price deviation from face value or principal there should be for discount bonds or premium bonds. This may be easily illustrated using a differential way to find the value of a bond.

Time	0	1	2	3	4	5	6	7
Cash Flows		$60	$60	$60	$60	$60	$60	$60+$1,000

First, let's think of a 7-year bond paying annual coupon interests of $60, i.e., the coupon interest rate is 6%. Assuming the required yield for this bond in the market is 6%, the traditional bond pricing equation above will yield a bond price of $1,000. By looking at the cash flows, it should be clear that investors are willing to pay $1,000 in this situation. The required yield of 6% means that investors want 6% annual interest payments when they, in a sense, lend their money to the issuer of the bond. If they are paid 6% interest every year, and get their principal back at the end of the bond contract, the present value of the bond should equal to principal or face value.

Next, let's think of a discount bond case, assuming the required yield is 7%. The bond price will be \$946.11 using the traditional bond pricing equation (1). In this case, the required yield of 7% means investors want 7% annual interest payments to lend \$1,000 to the bond issuer. If the bond has a 6% coupon interest, then the bond price should be adjusted due to the difference between required yield and coupon interest rate. We can see the price should be discounted but do not see by how much.

Intuitively, the bond price discount should be sum of the present values of under-paid coupon payments. Thus investors will calculate the present value of under-paid coupon interest payments and add (deduct since the value is negative) that from \$1,000. As a result of this adjustment, the bond price will be (\$1,000 + the sum of present values of under-paid coupon payments).

Cash flows in the time line below indicate under-paid coupon payments from the perspective of investors.

Time	0	1	2	3	4	5	6	7
Cash Flows		-\$10	-\$10	-\$10	-\$10	-\$10	-\$10	-\$10

Remember the investors' required yield is 7% but the bond pays 6%. Thus investors are under-paid by 1% (\$10 in dollar terms) every year. This means investors should extract the 1% under-paid coupon payments. Because the bond indenture sets the fixed coupon interest rate and face value, the adjustment can only be made in the bond price. Hence bond investors will require a discount in the bond price. Naturally, the appropriate discount amount will be sum of present values of under-paid coupons payments. In this example, the discount amount in the bond price will be -\$53.89 (sum of present values of \$10 for seven years at discounted at required yield of 7%). So the bond price will be \$1000 - \$53.89 = \$946.11.

Or, for the discount bond example:

Traditional Bond Pricing equation:

$$V_{bond} = \sum_{t=1}^{t=7} \frac{Coupon}{(1+y)^t} + \frac{Face\ Value}{(1+y)^7} = \sum_{t=1}^{t=7} \frac{\$60}{(1+0.07)^t} + \frac{\$1{,}000}{(1+0.07)^7} = \$946.11$$

Differential Bond Pricing equation:

$$V_{bond} = Face\ Value - \sum_{t=1}^{i=7} \frac{Under\ Paid\ Coupon}{(1+y)^t} = \$1{,}000 + \sum_{t=1}^{t=7} \frac{-\$10}{(1+0.07)^t} = \$1{,}000 - \$53.89 = \$946.11.$$

Now let's consider the premium bond case. Assume the bond pays 6% coupon interest payments each year. But investors think the appropriate required yield is 5%. Then investors receive higher interest payments than they require, compete for this bond and are willing to pay more than the face value. The question again is, how much more are they willing to pay? Following the same logic from above, the exact premium investors are willing to add to the face value will be the sum of present values of over-paid coupon interest payments. In this example, the over-paid annual coupon interest is \$10 in dollar terms.

Cash flows in the time line below indicate over-paid coupon payments from the perspective of investors.

Time	0	1	2	3	4	5	6	7
Cash Flows		\$10	\$10	\$10	\$10	\$10	\$10	\$10

Thus the exact premium investors are willing to add to the face value in determining the bond price will be \$57.86. Hence the bond price in this example will be \$1000 + \$57.86 = \$1,057.86.

Or, for the premium bond example:

Traditional Bond Pricing equation:

$$V_{bond} = \sum_{t=1}^{t=7} \frac{Coupon}{(1+y)^t} + \frac{FaceValue}{(1+y)^7} = \sum_{t=1}^{t=7} \frac{\$60}{(1+0.05)^t} + \frac{\$1{,}000}{(1+0.05)^7} = \$1{,}057.86$$

Differential Bond Pricing Equation:

$$V_{bond} = Face\,Value + \sum_{t=1}^{t=7} \frac{Over\ Paid\ Coupon}{(1+y)^t} = \$1{,}000 + \sum_{t=1}^{t=7} \frac{\$10}{(1+0.05)^t} = \$1{,}000 + \$57.86 = \$1{,}057.86.$$

More generally, this differential approach to value bonds can be described as follows:

$$V_{bond} = Face\,Value + \sum_{t=1}^{t=n} \frac{(C-y) * Face\,Value}{(1+y)^t}.$$

where n = time to maturity,
y = required yield,
Coupon = coupon (interest) rate (= C) * Face Value,
Face Value (= par value) = $1,000 in U.S.

Though purposely not identified as such, the relationship between required yield and price as discussed above is of course the financial bond pricing concept of duration (Macaulay 1938, Bierwag 1987). Again, as "Bond prices and yields are inversely related: as yields increase, bond prices fall; as yields fall, bond prices rise" (Bodie, Kane and Marcus 2009. pg. 514). Additional pedagogic in-class exercises with varying under-paid and over-paid coupons would reinforce the yield to maturity (required interest rate) inverse relationship and sensitivity of bond prices.

Bond Valuation Convexity

Recall, the second bond pricing observation is "An increase in a bond's yield to maturity results in a smaller price change than a decrease in yield of equal magnitude" (Bodie, Kane and Marcus 2009. pg. 514). This differential bond price impact from an increase (decrease) in yield to maturity refers to the well established financial bond pricing concept of convexity (Macaulay 1938, Bierwag 1987). Unfortunately, bond price convexity may have a greater intuitive pedagogic difficulty than the duration - bond price yield relationship (Brooks and Livingston 1992, Shirvani and Wilbratte. 2002, 2005).

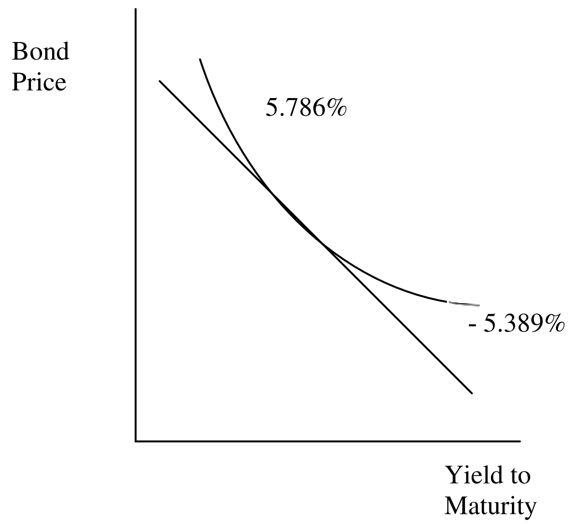
Fortunately, the suggested differential approach may also be used to demonstrate bond valuation convexity. The examples from above are shown in the Table 1: Results Summary below. The 1% increase in required yield reduced the bond price by $53.89 (i.e. 5.389% decrease in price), while the same 1% decrease in yield increased the bond price by $57.86 (i.e. 5.786% increase in price). Reviewing the respective differential bond pricing equations from above, students can easily see the same $10 coupon interest cash flow is discounted two different rates, respectively 7% and 5%.

Building on their prior time value of money knowledge, students can see the present value of the coupon cash flows at 7% will be smaller than the present value at 5%. An increase (decrease) in yield to maturity means a higher (lower) discount rate, thus the present value of the under- (over-) paid coupon interest cash flows must be smaller (greater). By emphasizing the present value of the coupon cash flows, the differential bond pricing approach offers an additional insight into the bond price valuation financial concept of convexity. The differential results are further demonstrated in Figure 1: Convexity - % Differential Bond Price Change.

Table 1
Results Summary

Required Yield	Traditional Approach Market Price – Face Value	Differential Approach Face Value +/– PV Payments	Percent Price Change
7% (yield increases)	$946.11- $1,000 = - $53.89	$1,000- $53.89 = $946.11	-5.389% = 53.89/1,000
5% (yield decreases)	$1,057.86- $1,000 = $57.86	$1,000+ $57.86 = $1,057.86	5.786% = 57.86/1,000

Figure 1:
Convexity - % Differential Bond Price Change



Conclusion

Teaching bond valuation with the differential approach using discounts and premiums has three pedagogic benefits. First, the bond price discount or premium is calculated directly rather than as a secondary calculation after the bond price is determined. Second, bond price valuation emphasizing discounts (under-paid) and premiums (over-paid) coupon interest reinforces the bond indenture fixed nature of the coupon interest rate and par value. Thus, under- (over-) paid coupons must be accounted for in the bond price, or changes in yields lead to changes in bond price, demonstrating the financial concept of duration or interest rate sensitivity. Third, the more intuitively difficult financial concept of the differential bond price response to increases (decreases) in yield to maturity, convexity, is shown as a straightforward result of using a higher (lower) discount rate for the under- (over-) paid coupon interest cash flow.

References

Berk, Jonathan and Peter DeMarzo. 2007. *Corporate Finance*. New York: Pearson/Addison Wesley.

Bierwag, Gerald O. 1987. *Duration Analysis: Managing Interest Rate Risk.* Cambridge: Ballinger.

Block, Stanley B., and Geoffrey A. Hirt. 2006. *Foundations of Financial Management.* New York: Irwin-McGraw-Hill.

Bodie, Zvi, Alex Kane, and Alan J. Marcus. 2009. *Investments,* New York: Irwin- McGraw-Hill.

Boyles, Gerald V., Thomas W. Secrest, and Robert B. Burney. 2005. "The Pricing of Bonds between Coupon Payments: From Theory to Market Practice." *Journal of Economics and Finance Education* 4(2): 61-68.

Brealey, Richard .A., Stewart C. Myers, and Allen J. Marcus. 2007. *Fundamentals of Corporate Finance.* New York: McGraw-Hill - Irwin.

Brigham, Eugene F. and Joel F. Houston. 2007. *Fundamentals of Financial Management.* Mason: Thomson South-Western.

Brooks, Robert and Miles Livingston. 1992. “Relative Impact of Duration and Convexity on Bond Price Changes”, *FPE Financial Practice and Education* 2(1), 93-99

Keown, Arthur, John. Martin, William. Petty, and David. Scott, Jr. 2006. *Foundations of Finance.* Boston: Pearson Education.

Macaulay, F. R. 1938. *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields, and Stock Prices in the United States since 1856*, New York: National Bureau of Economic Research.

Malkiel, Burton G. 1962. “Expectations, Bond Prices, and the Term Structure of Interest Rates,” *Quarterly Journal of Economics* 76: 197-218

Ross, Stephen A., Ross W. Westerfield, and Bradford D. Jordan. 2007. *Essentials of Corporate Finance.* New York: McGraw-Hill - Irwin.

Shirvani, Hassan, and Barry Wilbratte. 2002. "Two Pedagogical Simplifications of the Concept of Duration." *Journal of Economics and Finance Education* 1: 18-23.

Shirvani, Hassan, and Barry Wilbratte. 2005. " Duration and Bond Price Volatility: Some Further Results." *Journal of Economics and Finance Education* 4(1): 1-6.

Yang, Bill Z. 2002. “A Two-Dimensional Dual Presentation of Bond Market: A Geometric Analysis.” *Journal of Economics and Finance Education* 1(2): 11-17.

Appendix

Proof : The differential way yields the same bond pricing equation.

$$V_{bond} = Face\ \ Value - \sum_{t=1}^{t=n} \frac{(y - C) * Face\ \ Value}{(1+y)^t}$$

$$= Face\ \ Value * [1 - \sum_{t=1}^{t=n} \frac{(y-C)}{(1+y)^t}]$$

$$= Face\ \ Value * \left[1 - (y - C) \left(\frac{1 - \frac{1}{(1+y)^n}}{y} \right) \right]$$

$$= Face\ \ Value * \left[\frac{1}{(1+y)^n} + C \left(\frac{1 - \frac{1}{(1+y)^n}}{y} \right) \right]$$

$$= \frac{Face\ \ Value}{(1+y)^n} + Face\ \ Value * C \left(\frac{1 - \frac{1}{(1+y)^n}}{y} \right)$$

$$= \sum_{t=1}^{t=n} \frac{Coupon}{(1+y)^t} + \frac{Face\ \ Value}{(1+y)^n}$$

n = time to maturity,
y = required yield,
Coupon = coupon (interest) rate (= C) * Face Value,
Face Value (= par value) = $1,000 in U.S